Gauge-invariant Geometry of Space Curves: Application to Boundary Curves of Möbius-type Strips

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We derive gauge-invariant expressions for the twist Tw and the linking number Lk of a closed space curve, that are independent of the frame used to describe the curve, and hence characterize the intrinsic geometry of the curve. We are thus led to a *frame-independent* version of the Călugăreanu-White-Fuller theorem $Lk = Tw + Wr$ for a curve, where Wr is the writhe of the curve. The gauge-invariant twist and writhe are related to two types of geometric phases associated with the curve. As an application, we study the geometry of the boundary curves of closed twisted strips. Interestingly, the Möbius strip geometry is singled out by a characteristic maximum that appears in the geometric phases, at a certain critical width of the strip.

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Construct a Möbius strip (MS) by taking a long rectangular strip of paper, making a half-twist at one of its ends, and bringing the ends together. This strip has only one edge or boundary curve, which is a closed twisted space curve. Now, keeping the width constant, slide one of the ends on top of the other, so as to effectively decrease the length of the strip, and watch how the geometry of the boundary curve changes. Once the ratio of the width of the strip to its length increases to a certain value, a sharp twist appears on the boundary, which soon develops into a cone-like singularity. Since the boundary curve of the strip encodes information about this ratio, a key question is how the geometry of the curve changes with this parameter. Answering this is important in view of the fact that diverse systems, such as biopolymers[1], vortex filaments in a fluid[2], magnetic field lines[3], twisted optical fibers[4], and phase space trajectories in dynamical systems[5], can be modeled using closed twisted space curves. To this end, we first present a novel framework to describe the geometrical characteristics of a general twisted space curve, and then apply it to the boundary curve of the MS, as well as to other strips related to the MS, i. e., strips with several half-twists.

A basic concept in the study of the geometry of a space curve is that of a thin ribbon[6], one of whose edges is the given curve. To construct the other edge of the ribbon, one must first choose an orthogonal frame to describe the curve. A result which is widely applied is the celebrated Calŭgareanu-White- Fuller (CWF) theorem for a closed ribbon[6, 7, 8, 9]. It states that the integer linking number Lk of the two edges of the ribbon can be written as the sum of the twist Tw of the ribbon and the writhe Wr of the space curve. Tw characterizes the twist of the ribbon about its axis, while Wr is a measure of the non-planarity and non-sphericity[10] of the space curve.

Now, given a space curve, an infinite number of ribbons can be associated with it, owing to the inherent freedom in the choice of the orthogonal frame used. Each frame will yield a certain value for Tw, and hence for Lk. It is therefore desirable to find a way to define an *intrinsic* twist Tw_g of a curve about its axis, which would have a unique value, independent of the frame used. In this Letter, we achieve this by invoking the principle of gauge invariance. This leads us to the definition of an *intrinsic* linking number Lk_g of a space curve. In short, we obtain a ribbon-independent, gauge-invariant version of the CWF theorem for a closed space curve, given by

$$Lk_g = Tw_g + Wr. \qquad (1)$$

We also show that Tw_g can be viewed as a geometric phase or an anholonomy associated with the space curve. This classical geometric phase bears a close analogy to the quantum geometric phase obtained in the Aharonov-Anandan formulation[11]. Further, we show that Wr can also be interpreted as another distinct type of geometric phase, that can be associated with a certain *moving* space curve[12]. It follows that the linking number Lk_g of a closed space curve can be viewed as the difference between two distinct geometric phases associated with the curve.

Our analysis has a distinct advantage over the usual ribbon formulation[6] in the geometric characterization of curves with inflexion points, i. e., points where the curvature vanishes. Although curves with inflexion points are common in many applications, one assumes their absence in the standard formulation. In contrast, while Tw and Lk change discontinuously by unity whenever the curve develops an inflexion point, Tw_g and Lk_g remain unaffected. Hence the relationship (1) holds even when the curve has inflexion points, which is a very desirable feature in applications. To illustrate this, we present a systematic study of the change in geometry of the boundary curve of the Möbius strip, as well as those of other multi-twisted strips, as the strip-width varies. These curves typically acquire inflexion points at a certain critical strip-width.

We begin by developing a frame-independent formalism for describing the geometrical characteristics of a *general* closed space curve $C_1 = \mathbf{r}(s)$ of length L, parameterized by the arc length s. The unit tangent vector to the curve is $\mathbf{T}(s) = \mathbf{r}_s$ where the subscript s stands for the derivative with respect to s. Let $\mathbf{U}$ be a unit vector perpendicular to the curve, so that $\mathbf{T}, \mathbf{U}$ and $\mathbf{V} = \mathbf{T} \times \mathbf{U}$ define an orthogonal frame at every point on the curve. Since $\mathbf{U}$ must lie on the plane perpendicular to the tangent, there is an inherent freedom in choosing its direction on this plane, signifying a certain *gauge freedom* in its choice.

As mentioned in the beginning, the ribbon is bounded by $C_1 = \mathbf{r}(s)$ and a neighboring curve $C_2 = \mathbf{r}(s) + \epsilon \mathbf{U}(s)$, where ϵ is small. The twist Tw of the *ribbon* is defined[6] as $Tw(\mathbf{r}, \mathbf{U}) == (2\pi)^{-1} \int_0^L \mathbf{V} \cdot \mathbf{U}_s \, ds$. Hence Tw depends on the direction of $\mathbf{U}$ chosen, and is a frame-dependent quantity. Defining the complex unit vector $\mathbf{Q} = (\mathbf{U} + i\mathbf{V})/\sqrt{2}$, we get

$$Tw(\mathbf{r}, \mathbf{U}) = \frac{1}{2\pi} \int_0^L (\mathbf{T} \times \mathbf{U}) \cdot \mathbf{U}_s ds = \frac{i}{2\pi} \int_0^L \mathbf{Q}^* \cdot \mathbf{Q}_s ds.$$
(2)

Since we seek frame-independent geometrical characteristics of the curve, we make a gauge transformation $\mathbf{Q} \to \mathbf{q} = (\mathbf{u} + i\mathbf{v})/\sqrt{2} = \mathbf{Q} e^{i\eta(s)}$, which is essentially the rotation of the frame about the tangent, by an angle $\eta(s)$. It is easy to verify that $\mathbf{q}^* \cdot \mathbf{q}_s = \mathbf{Q}^* \cdot \mathbf{Q}_s + i \, \eta_s$. On integration, this leads to

$$i \int_0^L \mathbf{q}^* \cdot \mathbf{q}_s \, ds = i \int_0^L \mathbf{Q}^* \cdot \mathbf{Q}_s \, ds + [\eta(0) - \eta(L)]. \tag{3}$$

Since $[\eta(0) - \eta(L)] = \arg[\mathbf{q}^*(0) \cdot \mathbf{q}(L)] - \arg[\mathbf{Q}^*(0) \cdot \mathbf{Q}(L)]$, we see that

$$\begin{aligned} Tw_g(\mathbf{r}) &= \frac{i}{2\pi} \int_0^L \mathbf{Q}^* \cdot \mathbf{Q}_s \, ds - \frac{1}{2\pi} \arg[\mathbf{Q}^*(0) \cdot \mathbf{Q}(L)] \\ &= Tw(\mathbf{r}, \mathbf{U}) - \phi_T(\mathbf{r}, \mathbf{U}) \end{aligned} \tag{4}$$

is independent of $\mathbf{U}$, and may therefore be identified with the *gauge-invariant twist* of the curve. In Eq. (4), the second term $\phi_T(\mathbf{r}, \mathbf{U})$ is just the total phase (in units of 2π) accumulated by $\mathbf{Q}$ as the frame moves from 0 to L on the curve, and is given by

$$\phi_T(\mathbf{r}, \mathbf{U}) = \frac{1}{4\pi} \tan^{-1} \left(\frac{\mathbf{U}(L) \cdot \mathbf{V}(0) - \mathbf{V}(L) \cdot \mathbf{U}(0)}{\mathbf{U}(L) \cdot \mathbf{U}(0) + \mathbf{V}(L) \cdot \mathbf{V}(0)} \right).$$
(5)

Up to this point, the formulation is valid for both open and closed space curves. For the latter, the total phase ϕ_T is always an integer.

The twist Tw also appears in the following CWF theorem for a ribbon associated with a closed space curve $\mathbf{r}(s)$:

$$Lk(\mathbf{r}, \mathbf{U}) = Tw(\mathbf{r}, \mathbf{U}) + Wr(\mathbf{r}), \tag{6}$$

where the linking number $Lk(\mathbf{r}, \mathbf{U})$ of the curves C_1 and C_2 depends on the choice of $\mathbf{U}$, and Wr is the writhe of the curve. Following Dennis and Hannay[10], we find it convenient to cast the usual expression[6] for Wr in terms of the chord $\mathbf{C}$, as follows:

$$Wr = \frac{1}{4\pi} \int ds_1 \int ds_2 \left(\frac{\partial \mathbf{C}}{\partial s_1} \times \frac{\partial \mathbf{C}}{\partial s_2} \right) \cdot \mathbf{C}. \tag{7}$$

Here $\mathbf{C}(s_1, s_2) = [\mathbf{r}(s_1) - \mathbf{r}(s_2)]/|\mathbf{r}(s_1) - \mathbf{r}(s_2)|$ is a unit vector, and s_1, s_2 are two points on the given curve $\mathbf{r}(s)$. Thus Wr is a frame-independent, intrinsic property of the space curve, and is gauge-invariant.

Combining Eqs. (4) and (6), we obtain $Lk(\mathbf{r}, \mathbf{U}) - \phi_T(\mathbf{r}, \mathbf{U}) = Tw_g + Wr$. The gauge-invariance of Tw_g and Wr implies that

$$Lk_g(\mathbf{r}) = Lk(\mathbf{r}, \mathbf{U}) - \phi_T(\mathbf{r}, \mathbf{U}) \tag{8}$$

is independent of $\mathbf{U}$. Lk_g will be called the *gauge-invariant linking number* of the curve. Combining Eqs. (4) and (8), we obtain Eq. (1), the gauge-invariant form of the CWF theorem.

The advantage of the gauge-invariant formulation is that the expressions for Tw_g (and Lk_g) can now be written down in *any* frame. It is convenient to use the Frenet-Serret (FS) frame[13], where $\mathbf{U}$ and $\mathbf{V}$ are the principal normal $\mathbf{N} = \mathbf{T}_s/|\mathbf{T}_s|$ and the binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, respectively. The frame rotation is given by the FS equations, $\mathbf{T}_s = \kappa \mathbf{N}, \mathbf{N}_s = -\kappa \mathbf{T} + \tau \mathbf{B}$ and $\mathbf{B}_s = -\tau \mathbf{N}$, where $\kappa = |\mathbf{T}_s|$ is the curvature and $\tau = \mathbf{T} \cdot (\mathbf{T}_s \times \mathbf{T}_{ss})/(\mathbf{T}_s \cdot \mathbf{T}_s)$ is the torsion. Setting $\mathbf{U} = \mathbf{N}$ in Eqs. (2) and (5), and substituting into Eq. (4), we get

$$Tw_g = \frac{1}{2\pi} \int_0^L \tau \, ds - \phi_T(\mathbf{r}, \mathbf{N}). \tag{9}$$

In what follows, we show that Tw_g can be viewed as a geometric phase or an anholonomy associated with the space curve. Using the FS equations, we see that the first term in Eq. (9), i. e., the integrated torsion, is the rotation of the $(\mathbf{N}, \mathbf{B})$ plane about the tangent, as measured with respect to a local inertial (non-rotating) frame, as it moves once around the curve. This is called Fermi-Walker transport[14]. This term can be regarded as the *dynamical* phase accumulated by the $(\mathbf{N}, \mathbf{B})$ plane. The second term in Eq. (9) is obviously the corresponding *total* phase. Therefore, the gauge-invariant geometric phase ϕ_g associated with the $(\mathbf{N}, \mathbf{B})$ plane rotation on the curve is just the difference of the two phases above, i. e., $\phi_g = \phi_T(\mathbf{r}, \mathbf{N}) - (2\pi)^{-1} \int_0^L \tau \, ds$. On using Eq. (9), we get $\phi_g = -Tw_g$, showing that the gauge-invariant twist Tw_g is just the negative of the geometric phase ϕ_g (in units of 2π). This result is valid for both open and closed curves. We refer to this as tangent anholonomy, since the plane involved is perpendicular to the tangent.

Next, we show that the writhe Wr is related to another type of geometric phase. In Eq. (7) for Wr, the

unit chord $\mathbf{C}$ is a function of two independent variables, s_1 and s_2. Thus $\mathbf{C}$ can be regarded as a unit tangent vector to a certain space curve parameterized by s_1 for a given s_2, and which moves as s_2 varies. Therefore we can use the frame evolution formalism for a moving space curve[12]. Using the idea of Fermi-Walker transport once again, it can be shown that Wr is geometric phase associated with the rotation of the plane perpendicular to the chord $\mathbf{C}$. Hence Wr can be viewed as a chord anholonomy. These results show that Lk_g, which is an integer, is the difference between the chord anholonomy and the tangent anholonomy, which are not necessarily integers.

We now apply the gauge-invariant CWF theorem to study the geometry of the boundary curves of twisted closed strips. These strips are surfaces described by

$$
\begin{aligned}
x &= \left(R + w\cos\frac{nt}{2}\right)\cos t, \ y = \left(R + w\cos\frac{nt}{2}\right)\sin t, \\
z &= w\sin\frac{nt}{2},
\end{aligned}
\tag{10}
$$

with parameters $-\alpha \leq w \leq \alpha$ and $0 \leq t \leq 2\pi$. The length of the strip is $2\pi R$, and we set $R = 1$ for convenience. α is the half-width of the strip, and the integer n is the number of half-twists on the strip (so that $n = 1$ refers to the well-known Möbius strip). The boundary of the strip corresponds to $w = \pm\alpha$ in Eq. (10). For odd n, the strip is a non-orientable surface with one boundary curve, so that the range of t is $[0, 4\pi]$. For even n, the strip is orientable, with two boundary curves with similar geometries, and the range of t is $[0, 2\pi]$ for each curve. Topologically, for all widths, the boundary curves of the odd-n strips with $n > 1$ are knotted curves (e. g, it is a trefoil knot for $n = 3$, a five-pointed star knot for $n = 5$, etc.), while those of the even-n strips are not knotted. The boundary curve of the MS clearly does not fall in either of these classes, since it is the only case in which the boundary curve of a non-orientable strip is not a knot. For all n, the geometry of the boundary curve of the strip has a non-trivial dependence on its width, as we shall show explicitly.

Symbolic manipulation[15] facilitates the analytic calculation of curvature and torsion of the boundary curves of the strips given by Eq. (10). For each n, when the half-width attains a critical value $\alpha_c == (1 + \frac{1}{4}n^2)^{-1}$, the curvature vanishes at n points on the boundary curve[16]. For the MS, the critical boundary curve, corresponding to a half-width $\alpha_c = \frac{4}{5}$, has a single inflexion point (labeled by $t = t_c$). Plotting the boundary curves of the MS with various half-widths in Fig. 1, we see that the critical curve at the above α_c is distinguished by a local 'straightening' of the curve around the point $t_c = 2\pi$.

A Taylor expansion of $\kappa(t)$ and $\tau(t)$ near α_c and t_c gives $\kappa \simeq [(\alpha - \alpha_c)^2 + 9b^2v_c^2(t - t_c)^2]^{\frac{1}{2}}v_c^{3/2}$ and $\tau \simeq -3b(\alpha - \alpha_c)/[(\alpha - \alpha_c)^2 + 9b^2v_c^2(t - t_c)^2]$, where $b = \frac{6}{5}$ and $v_c = |d\mathbf{r}/dt|_c = 1/\sqrt{5}$. These expressions show that at $t = t_c$,
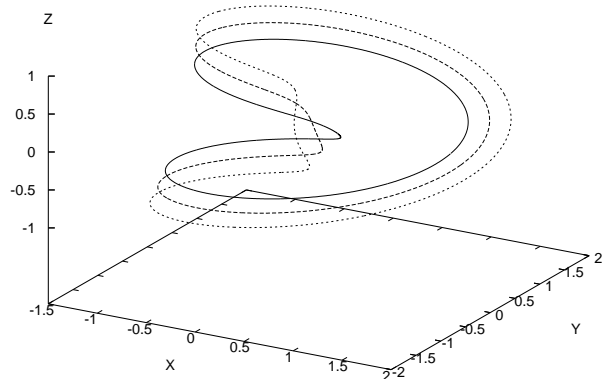

FIG. 1: Boundary curves of Möbius strips with half-widths $\alpha = 0.6, 0.8$ and 1.0. The middle curve is the critical curve with $\alpha = 0.8$, and the local straightening of this curve near the inflexion point is clearly visible.

τ is singular at α_c, and changes sign as the parameter α passes through α_c. We point out that the functional form of κ and τ near the inflexion point is identical to that obtained in an earlier study[3], which investigated the behavior of a curve in the vicinity of an inflexion point, the only difference being the value of b, which was unity for the illustrative example of the curve chosen there.

To find the effect of the above singularity in τ, we calculate the angle of twist of the $(\mathbf{N}, \mathbf{B})$ plane over an interval $2t_0$ centered about t_c: this is $\int_{-t_0+t_c}^{t_0+t_c} \tau v \, dt = \tan^{-1}\left(t_0/[bv_c(\alpha - \alpha_c)]\right)$. As we pass through α_c, this angle changes from $-\pi$ to π (irrespective of the value of t_0), giving rise to a jump of 2π, as discussed in Ref. [3].

Interestingly, the above jump in the integrated torsion resulting from an inflexion point is accompanied by an identical jump in $\phi_T(\mathbf{r}, \mathbf{N})$. A Taylor expansion of $\mathbf{N}$ near inflexion point yields $\mathbf{N} \simeq \kappa^{-1}[a_1(t - t_c), a_2(\alpha - \alpha_c), a_3(\alpha - \alpha_c)]$, where a_i $(i = 1, 2, 3)$ are constants. This shows that at $t = t_c$ for $\alpha \to \alpha_c - 0$, $\mathbf{N}$ rotates by π about the tangent, while for $\alpha \to \alpha_c + 0$, it rotates by $-\pi$. The same is also true for $\mathbf{B}$. Therefore, the angle of rotation of the $(\mathbf{N}, \mathbf{B})$ plane increases by 2π as we pass the inflexion point. As a result, Tw_g computed from Eq. (9) does not show a jump if the curve develops an inflexion point. The same behavior holds good for Lk_g.

Next, we report the results for the dependence of Wr and Tw_g of the boundary curves of the closed twisted strips (Eq. (10)) on the strip width. Wr of the boundary curve was directly computed from Eq. (7) using numerical integration. In Fig. 2, we plot the dependence of the writhe Wr of the boundary curves of various twisted strips on the dimensionless parameter $a = (\alpha - \alpha_c)/\alpha_c$. We find that for the boundary curves of infinitesimally

thin narrow-width strips with $\alpha \simeq 0$, $Wr = n$ for all odd n, while $Wr = 0$ for all even n. We have therefore plotted, for convenience, a shifted writhe ($Wr - n$) for $n = 1, 3, 5, 7$. For $n = 2, 4, 6$, Wr has been plotted without such a shift.

As seen in Fig. 2, Wr does not undergo jumps, and remains smooth as the boundary curve develops inflexion points at a corresponding critical width. The variation of
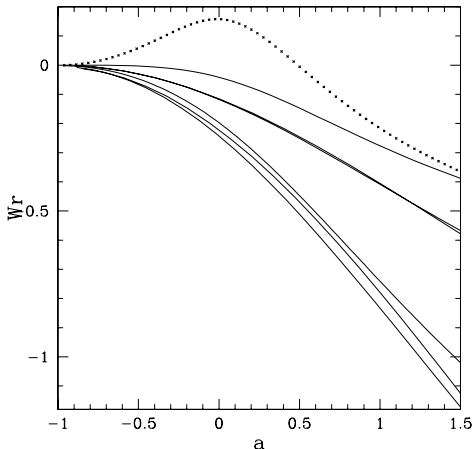

FIG. 2: Dependence of Wr of boundary curves of various n-twisted strips (shifted down by n for all odd n) on $a = (\alpha - \alpha_c)/\alpha_c$. The top curve corresponds to the Möbius strip with $n = 1$, followed by three curves that correspond to $n = 2, 4, 6$ (top to bottom). The three other curves below these correspond to $n = 3, 5, 7$.

Wr with strip-width is significantly larger for the odd n, non-orientable class of strips, as compared to the even-n, orientable class. Another notable feature is that the MS boundary curve gets singled out due to its characteristic *maximum* near the critical width. This is a reflection of the fact that the MS boundary curve with $n = 1$ is indeed special, as it belongs to a class distinct from the rest, as we have discussed above.

Analytical expressions for τ, $\mathbf{N}$ and $\mathbf{B}$ were used to calculate Tw_g from Eq. (9). The variation of Tw_g with a for various n values turns out to be essentially complementary to the variation of Wr given in Fig. 2, and hence is not given here. It is sufficient to note that Tw_g for the MS boundary curve has a characteristic *minimum*, and so the geometric phase $\phi_g = -Tw_g$ has a *maximum*. By adding Wr and Tw_g, we find that $Lk_g = (n + 2)$ for odd values of n, while $Lk_g = 1$ for even values of n.

The general CFW theorem Eq. (1) relates topological and geometric properties of a space curve, and would therefore find many applications in physics and biology. For instance, the two boundary curves of an orientable twisted strip can represent the two strands of a closed DNA[1], the distance between them being the strip width. Our results may also find application in the fabrication of biomaterials using molecular architecture[17], where information about the conformational changes in the strands caused by the variation of width-to-length ratios, number of twists, etc., would be useful.

Since the geometric phase describes the change in the polarization of light[4] as it propagates along a nonplanar circuit, our results on the dependence of the geometric phase ϕ_g on the width-to-length ratio of the strip can be exploited to tune polarizations. This can be achieved by fabricating an optical fiber glued to the boundary of an opaque twisted strip, and studying light propagation through the fiber.

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[1] See, for instance, M. D. Frank-Kamanetskii and A. V. Vologodski, Sov. Phys. Usp. **24**, 679 (1981); K. A. Hoffman, R. S. Manning and J. H. Maddocks, Biopolymers **70**, 145 (2003).
[2] H. Hasimoto, J. Fluid Mech. **51**, 477 (1972).
[3] H. K. Moffatt and R. L. Ricca, Proc. R. Soc. Lond. A **439**, 411 (1992); R. L. Ricca, J. Phys. A: Math. Gen. **28**, 2335 (1995).
[4] R. Y. Chiao and Y.-S. Wu, Phys. Rev. Lett. **57**, 933 (1986).
[5] Radha Balakrishnan and Indubala I. Satija, Phys. Lett. A **335**, 20 (2005).
[6] F. B. Fuller, Proc. Nat. Acad. Sci. USA **68**, 815 (1971); *ibid.* **75**, 3557 (1978).
[7] G. Călugăreanu, Rev. Math. Pures Appl. **4**, 5 (1959); Czech. Math. J. **11**, 588 (1961).
[8] J. H. White, Am. J. Math. **91**, 693 (1969).
[9] W. F. Pohl, J. Math. Mech. **17**, 975 (1968).
[10] M. R. Dennis and J. H. Hannay, preprint, math-ph/0503012 (2005).
[11] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987); N. Mukunda and R. Simon, Ann. Phys. **228**, 205 (1993).
[12] Radha Balakrishnan, A. R. Bishop, and R. Dandoloff, Phys. Rev. Lett. **64**, 2107 (1990); Phys. Rev. B **47**, 3108 (1993); *ibid.* **47**, 5438 (1993).
[13] D. J. Struik, *Lectures on Classical Differential Geometry*, (Addison-Wesley, Reading, Mass., 1961).
[14] M. Kugler and S. Shtrikman, Phys. Rev. D **37**, 934 (1988); R. Dandoloff and W. J. Zakrzewski, J. Phys. A: Math. Gen. **22**, L469 (1989).
[15] S. Wolfram, *The Mathematica Book*, 3rd edition (Wolfram Media/Cambridge University Press, Cambridge, 1996).
[16] A similar result has been found for helices wound around a torus, in E. Radescu and G. Vaman, Phys. Rev. E **64**, 056609 (2001).
[17] S. Zhang, Nature Biotechnology **21**, 1173 (2003).